Hodge Theory Lecture 2

Mitchell Gaudet

March 2, 2025

Supports

Consider an open subset $U \subseteq \mathbb{R}^n$ and a continuous function $f : U \to \mathbb{R}$. We can speak about the set $\{x \in U : f(x) \neq 0\}$, and in particular can talk about its closure in U, the support of f.

The usage of this comes in large part when the support is a compact set, then f has most of the nice properties of continuous functions from compact sets.

There is, however, a problem when defining compactly supported L^p functions. The problem is that they are only an **equivalence class** of functions, and so the support, as defined before, is not well-defined.

However, we may reinterpret the support as the complement of an open set, the largest open set where f is zero. This leads us to even defining supports for distributions.

Definition 0.1. Given a distribution $\phi : C_c^\infty(U) \to \mathbb{R}$, we define its **Support** as

$$supp(\phi) = \bigcup_{V \in presupp(\phi)} U$$

where $presupp(\phi)$ is the collection of open subsets $V \subseteq U$ such that $\phi(f) = 0$

for each $f \in C_c^\infty(U)$ with $supp(f) \subseteq V$.

Given a function $g \in L^p(U)$, or continuous, or smooth, define that support as the support of ϕ_g(see the problem set 1).

Exercise 0.1. Show this definition agrees with the usual one when g is continuous.

This is, is some sense, the right way to think about the support of distributions, since their behaviour is defined by their actions on functions on open sets, and so the support should be defined in terms of open sets.

Mollifiers(Corrected)

Suppose that $U \subseteq \mathbb{R}^n$ is open and $K \subseteq U$ is compact, and define $W^{k,2}_{1,K} = \{f \in W^{k,2}(U) : supp(f) \subseteq K\}$. Then we have

Definition 0.2. There exists $t > 0$ and a family $\{F_\varepsilon\}_{\varepsilon \in [0,t)}$ of linear operators $W^{k,2}_{1,K} \to C_c^\infty(U, \mathbb{R}) \cap W^{k,2}(U)$ with the following properties:

1. F_ε is bounded uniformly from $W^{k,2}_{1,K} \to W^{k,2}(U)$, where $W^{k,2}_{1,K}$ has the induced norm.

2. For any operator of the form

$$(Pf)(x) = \sum_{\alpha=1}^{m} \sum_{i=1}^{n} a_i^\alpha(x) \frac{\partial f_\alpha}{\partial x^i}(x) + \sum_{\beta=1}^{m} b^\beta(x) f_\beta(x)$$

   where $a_i^\alpha, b^\beta \in C^\infty(\overline{U})$ it holds that $F_\varepsilon P - PF_\varepsilon$ is a uniformly bounded operator $W^{k,2}_{1,K} \to W^{k,2}(U)$. (Here α is a number, not a multi-index)

3. $\lim_{\varepsilon \to 0} \|F_\varepsilon u - u\| = 0$ for any $u \in W^{k,2}_{1,K}$

4. $supp(F_\varepsilon u) \subseteq \{x \in U : d(x, K) \leq \varepsilon\}$ for any $u \in W^{k,2}_{1,K}$

We call such a family a **mollifier.** Of note is that t depends crucially on K. In particular, $t < dist(K, \partial U)$.

We can also find similar results for $W^{k,2}_{m,K} = \{f \in W^{k,2}_m(U) : supp(f^i) \subseteq K, \forall i = 1, ..., m\}$

A note about terminology: I will often use $H^1 = W^{1,2}$.

Cohomology

Consider the following problem: You would like to figure out how to rigorously define what a "hole" in $\mathbb{R}^2$ is. Roughly speaking, it should be a disk that cannot be filled in fully.

Such a disk should be represented as a circle that isn't the boundary of the interior disk. So we should define a (two-dimensional) hole as a closed curve S such that $S \neq \partial D$ for any open set D.

While there is a way to make this completely rigorous, we note the following, if $S = \partial D$ then for any closed 1-form ω it holds that

$$\int_S \omega = \int_{\partial D} \omega = \int_D d\omega = 0$$

On the other hand, note that if

$$c_1, c_2$$

are two paths with the same start and end-points, then we should have

$$\int_{c_1} \omega = \int_{c_2} \omega$$

if either the loop formed by c_1, c_2 contains no holes, or the form is exact.

We also see that there is some sense of duality between ∂, d. In particular,

$L(S)(\omega) = \int_S \omega$ has

$$L(\partial D)(\omega) = L(D)(d\omega)$$

and so one might expect closed but not exact forms to represent holes, especially when considering the usual form $d\theta$ on $\mathbb{R}^2 - \{0\}$. In many cases, this is true! There is however, a much better way of thinking about cohomology, especially in geometric analysis: as obstructions to piecing together local data to global.